Concept Development Lessons

How can I help students develop a deeper understanding of Mathematics?

A PROFESSIONAL DEVELOPMENT MODULE

Introduction

The Formative Assessment Lessons are of two types; those that focus on the development of conceptual understanding and those that focus on problem solving. *Concept Development* lessons are intended to assess and develop students' understanding of fundamental concepts through activities that engage them in classifying and defining, representing concepts in multiple ways, testing and challenging common misconceptions, and exploring structure. *Problem Solving* lessons are intended to assess and develop students' capacity to *select* and deploy their mathematical knowledge in non-routine contexts and typically involve students in comparing and critiquing alternative *approaches* to solving a problem.

In this PD module, we focus on *Concept Development* lessons. Research has shown that individual, routine practice on standard problems does little to help students deepen their understanding of mathematical concepts. Teaching becomes more effective when existing interpretations (and misinterpretations) of concepts are shared and systematically explored within the classroom. The lessons described here typically begin with a formative assessment task that exposes students' existing ways of thinking. The teacher is then offered specific suggestions on how these may be challenged and developed through collaborative activities. New ideas are constructed through reflective discussion. This process places considerable pedagogic demands on teachers, and it is these demands that this module is intended to explore.

Activities

Acknowledgement:
Parts of this material, including the video, were adapted from *Improving Learning in Mathematics*, a government funded program in the UK. See: Swan, M; (2005). *Improving Learning in Mathematics*, challenges and strategies, Department for Education and Skills Standards Unit. Obtainable in the UK from http://tlp.excellencegateway.org.uk/pdf/Improving_learning_in_maths.pdf

Activity A: Using the assessment tasks

Time needed: 30 minutes.

Each *Formative Assessment Lesson* is preceded by an introductory assessment task. The purpose of this is to discover the interpretations and understandings that students bring to this particular area of mathematical content. This task is given to individual students a day or more before the main lesson and the information gathered from student responses are then used to plan and direct the lesson.

In this activity, participants begin to look at a selection of such assessment tasks and consider the kind of information they provide, and how best to respond to students.

The examples used below are taken from the following lessons:

- Interpreting distance-time graphs (Middle School)
- Increasing and decreasing quantities by a percent (Middle School)
- Interpreting algebraic expressions (High School)

It would be helpful if participants could see the complete materials[1] for one of these.

Look at the assessment tasks from three lessons on **Handout 1**.

- Try to anticipate the kinds of mistakes your students would make on each of these tasks.
- What common difficulties would you expect?

Now look at the samples of student work on **Handout 2**.

- What does the student appear to understand? Where is your evidence?
- List the errors and difficulties that are revealed by each response.
- Try to identify the thinking that lies behind each error.
- What feedback would you give to each student? Write down your comments on the work.

Following each assessment task, we have provided suggestions for follow-up questions that would move students' thinking forward. These are given on **Handout 3.** Compare the feedback you have written to these questions.

- Do you normally give feedback to students in the form of questions?
- What are the advantages of using questions rather than more directive guidance?
- Can you suggest better questions to the ones provided?

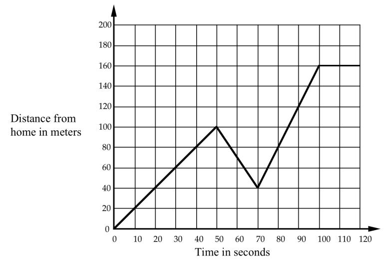
Research has shown that giving students scores or grades on their work is counter-productive, and this should not be done with these assessment tasks. This is discussed in more detail in Professional Development Module 1, 'Formative Assessment.'

[1] Available from http://map.mathshell.org.uk/materials/lessons.php

Handout 1: Assessment tasks

Journey to the Bus Stop

Every morning Tom walks along a straight road from his home to a bus stop, a distance of 160 meters. The graph shows his journey on one particular day.



1. Describe what may have happened.
   You should include details like how fast he walked.

   ....

2. Are all sections of the graph realistic? Fully explain your answer.

   ....

Handout 2: Sample student work

Jodie's response

Tom walked along a road for 100metres instead of walking another 30metres he took a short cut down an alleyway which took he 20 minutes he walked very quickly then he caught the bus to his college which took about 50 minutes

Maxine's response

when he get out he starts walking fast to the bus stop then he slows down the he picks up the speed again and then ~~the~~ this speed goes ~~cont~~ constant

Handout 3: Sample follow-up questions

Common issues:	Suggested questions and prompts:
Student interprets the graph as a picture For example: The student assumes that as the graph goes up and down, Tom's path is going up and down. Or: The student assumes that a straight line on a graph means that the motion is along a straight path. Or: The student thinks the negative slope means Tom has taken a detour.	• *If a person walked in a circle around their home, what would the graph look like?* • *If a person walked at a steady speed up and down a hill, directly away from home, what would the graph look like?* • *In each section of his journey, is Tom's speed steady or is it changing? How do you know?* • *How can you figure out Tom's speed in each section of the journey?*
Student interprets graph as speed–time The student has interpreted a positive slope as speeding up and a negative slope as slowing down.	• *If a person walked for a mile at a steady speed, away from home, then turned round and walked back home at the same steady speed, what would the graph look like?* • *How does the distance change during the second section of Tom's journey? What does this mean?* • *How does the distance change during the last section of Tom's journey? What does this mean?* • *How can you tell if Tom is traveling away from or towards home?*

Activity B: What causes mistakes and misconceptions?

Time needed: 15 minutes.

This activity is intended to encourage teachers to see that student errors may be due to deep-rooted misconceptions that should be exposed and discussed in classrooms.

- Why do students make mistakes in Mathematics?
- What different types of mistakes are there? What are their causes?
- How do you respond to each different type? Why?

Draw out the different possible causes of mistakes. These may be due to due to lapses in concentration, hasty reasoning, memory overload or a failure to notice important features of a problem. Other mistakes, however, may be symptoms of alternative ways of reasoning. Such 'misconceptions' should not be dismissed as 'wrong thinking' as they may be *necessary* stages of conceptual development.

Consider generalizations commonly made by students, shown on **Handout 4.**

- Can you contribute some more examples to this list?
- Can you think of any misconceptions you have had at some time?
- How were these overcome?

Many 'misconceptions' are the results of students making generalizations from limited domains. For example, when younger children deal solely with natural numbers they infer that 'when you multiply by ten you just add a zero.' Later on, this leads to errors such as 3.4 x10 = 3.40.

- For what domains do the following generalizations work? When do they become invalid?
    - If I subtract something from 12, the answer will be smaller than 12.
    - The square root of a number is smaller than the number.
    - All numbers may be written as proper or improper fractions.
    - The order in which you multiply does not matter.

- Can you think of other generalizations that are only true for limited domains?

- There are two common ways of reacting to pupils' errors and misconceptions:
  Avoid them whenever possible: "If I warn pupils about the misconceptions as I teach, they are less likely to happen. Prevention is better than cure."
  Use them as learning opportunities: "I actively encourage learners to make mistakes and learn from them."
  What are your views?

- Discuss the principles given on **Handout 5**. This describes the advice given in the research. How do participants feel about this advice?

Handout 4: Generalizations commonly made by students

What other examples can you add to this list?
Can you think of any misconceptions you have had at some time?
How were these overcome?

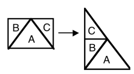
0.567>0.85
The more digits a number has, the larger is its value.

3÷6=2
You always divide the larger number by the smaller one.

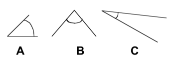
0.4>0.62
The fewer the number of digits after the decimal point, the larger is its value.
It's like fractions.

5.62 x 0.65 > 5.62
Multiplication always makes numbers bigger.

1 litre costs $2.60; 4.2 litres cost $2.60 x 4.2; 0.22 litres cost $2.60 ÷ 0.22
If you change the numbers in a question, you change the operation you have to do.



Area of rectangle ≠ Area of triangle
If you dissect a shape and rearrange the pieces, you change the area.



Angle A is greatest. Angle C is greatest.
The size of an angle is related to the size of the arc or the length of the arms of the angle.

If x+4<10, then x=5.
Letters represent particular numbers'.

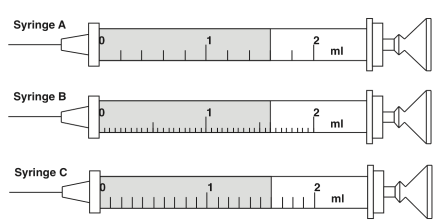
3+4 = 7+2 = 9+5 = 14
Equals' means 'makes'.

In three rolls of a die, it is harder to get 6,6,6 than 2,4,6.
Special outcomes are less likely than more representative outcomes.

Handout 5: Principles to discuss

These principles are backed up by research evidence.
Discuss the implications for your own teaching.

- Teaching approaches that encourage the exploration of misconceptions through discussions result in deeper, longer-term learning than approaches that try to avoid mistakes by explaining the 'right way' to see things from the start.
- It is helpful if discussions focus on known difficulties. Rather than posing many questions in one session, it is better to focus on a challenging question and encourage a variety of interpretations to emerge, so that learners can compare and evaluate their ideas.
- Questions can be juxtaposed in ways that create a tension (sometimes called a 'cognitive conflict') that needs resolving. Contradictions arising from conflicting methods or opinions can create an awareness that something needs to be learned. For example, asking learners to say how much medicine is in each of the following syringes may result in answers such as "1.3ml, 1.12ml and 1.6ml". "But these quantities are all the same!" This provides a start for a useful discussion on the denary nature of decimal notation.



- Activities should provide opportunities for meaningful feedback. This does not mean providing summative information, such as the number of correct or incorrect answers. More helpful feedback is provided when learners compare results obtained from alternative methods until they realise why they get different answers.
- Sessions include time for whole group discussion in which new ideas and concepts are allowed to emerge. This requires sensitivity so that learners are encouraged to share tentative ideas in a non-threatening environment.
- Opportunities should be provided for learners to 'consolidate' what has been learned through the application of the newly constructed concept.

Activity C: The Formative assessment lesson.

Time needed: 20 minutes.

This activity is designed to help participants recognize some of the broad principles upon which the Formative Assessment Lessons are designed.

Choose one of the Formative assessment lessons that relates to the assessment tasks you looked at in Activity 1:

- Interpreting distance-time graphs
- Increasing and decreasing quantities by a percent
- Interpreting algebraic expressions

- Compare the lesson with the broad structure described on **Handout 6**.
- How far does the lesson follow this structure and where does it deviate?
- Can you see reasons why the lessons have been designed in this way?
  Try to describe these reasons.
- Which type of activity has been used in the main activity? (See **Handout 7**)
- Work on the main activity together.
- How does the main activity address the common conceptual issues that you considered in the student work on **Handout 2**?
- How are the students expected to learn from the activity?

On the following pages, we further describe and illustrate each of the four task genres shown on **Handout 7.**

Handout 6: Structure of the Concept lessons

Broadly speaking, each Concept Formative Assessment Lesson is structured in the following way, with some variation, depending on the topic and task:

- **(Before the lesson) Students complete an assessment task individually**
  This assessment task is designed to clarify students' existing understandings of the concepts under study. The teacher assesses a sample of these and plans appropriate questions that will move student thinking forward. These questions are then introduced in the lesson at appropriate points.

- **Whole class introduction**
  Each lesson begins with the teacher presenting a problem for class discussion. The aim here is to intrigue students, provoke discussion and/or model reasoning,

- **Collaborative work on a substantial activity**
  At this point the main activity is introduced. This activity is designed to be a rich, collaborative learning experience. It is both accessible and challenging; having multiple entry points and multiple solution paths. It is usually done with shared resources and is presented on a poster.
  Four types of activity are commonly used as shown in **Handout 6**. Students are involved in:

  - classifying mathematical objects & challenging definitions
  - interpreting multiple representations
  - evaluating conjectures and assertions
  - modifying situations & exploring their structure

  These will be explored more fully later in this module. It is not necessary for every student to complete the activity. Rather we hope that students will come to understand the concepts more clearly.

- **Students share their thinking with the whole class**
  Students now share some of their learning with other students. It is through explaining that students begin to clarify their own thinking. The teacher may then ask further questions to provoke deeper reflection.

- **Students revisit the assessment task**
  Finally, students are asked to look again at their original answers to the assessment task. They are either asked to improve their responses or are asked to complete a similar task. This helps both the teacher and the student to realize what has been learned from the lesson.

Handout 7: Genres of activity used in the Concept lessons

The main activities in the concept lessons are built around the following four genres. Each of these types of activity is designed to provoke students to reason in different ways; to recognize properties, to define, to represent, to challenge conjectures and misconceptions, to recognize deeper structures in problems.

1. **Classifying mathematical objects**
   Mathematics is full of conceptual 'objects' such as numbers, shapes, and functions. In this type of activity, students examine objects carefully, and classify them according to their different attributes. Students have to select an object, discriminate between that object and other similar objects (what is the same and what is different?) and create and use categories to build definitions. This type of activity is therefore powerful in helping students understand different mathematical terms and symbols, and the process by which they are developed.

2. **Interpreting multiple representations**
   Mathematical concepts have many representations; words, diagrams, algebraic symbols, tables, graphs and so forth. These activities allow different representations to be shared, interpreted, compared and grouped in ways that allow students to construct meanings and links between the underlying concepts.

3. **Evaluating mathematical statements**
   These activities offer students a number of mathematical statements or generalizations. These statements may typically arise from student misconceptions, for example: "The square root of a number is smaller than the number". Students are asked to decide on their validity and give explanations for their decisions. Explanations usually involve generating examples and counterexamples to support or refute the statements. In addition, students may be invited to add conditions or otherwise revise the statements so that they become 'always true'.

4. **Exploring the structure of problems**
   In this type of activity, students are given the task of devising their own mathematical problems. They try to devise problems that are both challenging and that they know they can solve correctly. Students first solve their own problems and then challenge other students to solve them. During this process, they offer support and act as 'teachers' when the problem solver becomes stuck. Creating and solving problems may also be used to illustrate doing and undoing processes in mathematics. For example, one student might draw a circle and calculate its area. This student is then asked to pass the result to a neighbor, who must now try to reconstruct the circle from the given area. Both students then collaborate to see where mistakes have arisen.

Activity D: Classifying mathematical objects

Time needed: 20 minutes.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. (Common Core State Standards, p.7)

Understanding a concept involves four mental processes: bringing it to the foreground of attention, naming and describing its properties (identifying); identifying similarities and differences between this concept and others (discriminating); identifying general properties of the concept in particular cases of it (generalizing); and perceiving a unifying principle (synthesizing, defining) (Sierpinska, 1994).

The following examples illustrate these practices

- Work on some of the activities on **Handout 8**. Here the objects are geometrical shapes. The objects could equally well be equations, words, numbers…
- What kinds of 'objects' do you ask students to classify and define in your classroom?
- Try to develop an activity using one of these types for use in your own classroom. Try to devise examples that force students to observe the properties of objects carefully, and that will create discussion about definitions.

  Try out your activity and report back on it in a later session.

The types of activity shown here may be extended to almost any context. The objects being described, defined and classified could be numerical, geometric or algebraic.

Similarities and differences
Students may, for example, decide that the square is the odd one out because it has a different perimeter to the other shapes (which both have the same perimeter); the rectangle is the odd one out because it has a different area to the others and so on. Properties considered may include area, perimeter, symmetry, angle, convexity etc. Participants should try to devise their own examples.

Properties and definitions
None of the properties by themselves defines the square. It is interesting to consider what other shapes are included if just one property is taken. For example, when the property is ‘Two equal diagonals’ then all rectangles and isosceles trapezoids are included - but is that all the cases?
Taken two at a time, then results are not so obvious. For example, ‘four equal sides’ and ‘four right angles’ defines a square, but ‘diagonals meet at right angles’ and ‘four equal sides’ does not (what else could this be?).

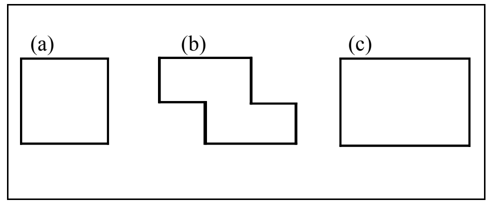
Creating and testing definitions
Participants usually write a rather vague definition of ‘polygon’ to begin with, such as: "A shape with straight edges." They then see that this is inadequate for the given examples. This causes them to redefine more rigorously, like "a plane figure that is bounded by a closed path or circuit, composed of a finite sequence of straight line segments." Defining is difficult, and students should realize that there are competing definitions for the same idea (such as ‘dimension’, for example).

Classifying using two-way tables
Two-way tables are not the only representation that may be used, of course, and participants may suggest others. Venn diagrams and tree diagrams are just two examples.

Handout 8: Classifying mathematical objects

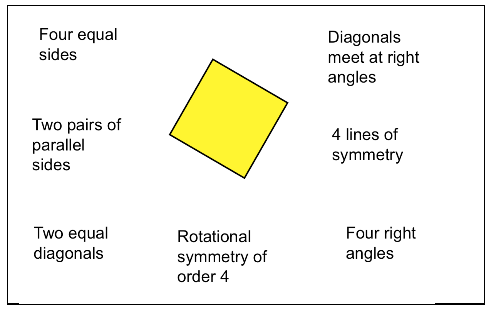
Similarities and differences

Show students three objects.
"Which is the odd one out?"
"Describe properties that two share that the third does not."
"Now choose a different object from the three and justify it as the odd one out."



Properties and definitions

Show students an object.
"Look at this object and write down all its properties."
"Does any *single* property constitute a *definition* of the object? If not, what other object has that property?"
"Which *pairs* of properties constitute a definition and which pairs do not?"

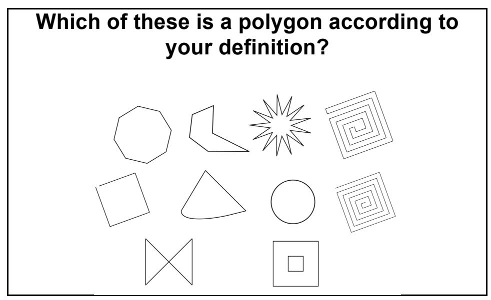


Creating and testing a definition

Ask students to write down the definition of a polygon, or some other mathematical word.

"Exchange definitions and try to improve them."

Show students a collection of objects.
"Use your definition to sort the objects."
"Now improve your definitions."



Classifying using a two-way table

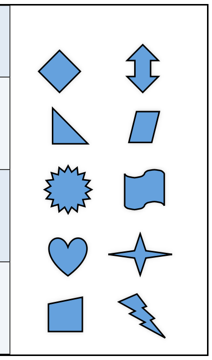
Give students a two-way table to sort a collection of objects.

"Create your own objects and add these to the table."

"Try to justify why particular entries are impossible to fill."

“Classify the objects according to your own categories.
Hide your category headings.
Can your partner identify the headings from the way you have sorted the objects?”

	No rotational symmetry	Rotational symmetry
No lines of symmetry		
One or two lines of symmetry		
More than two lines of symmetry		



Activity E: Interpreting multiple representations

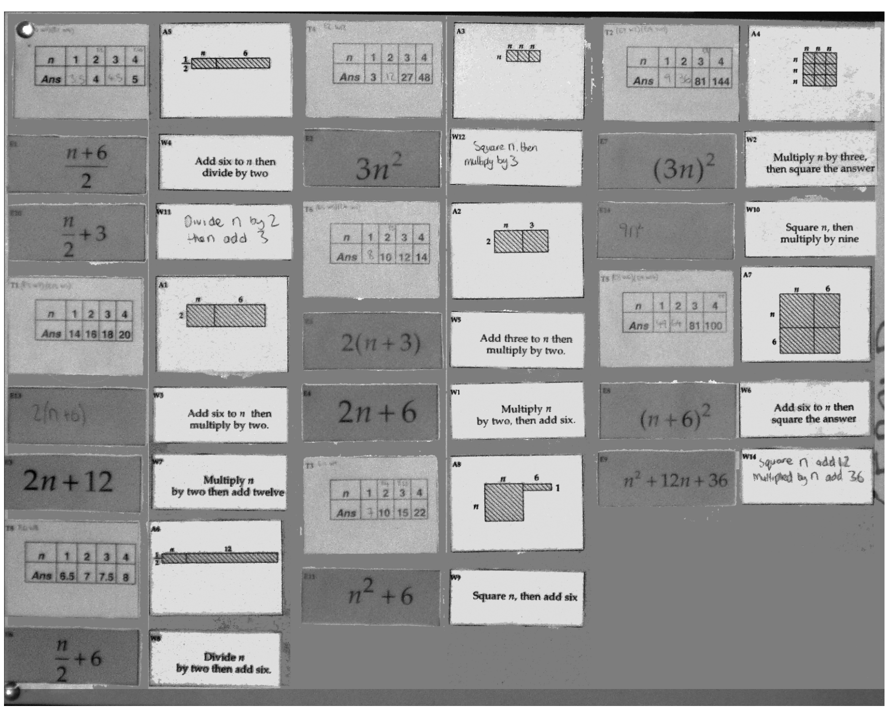
Time needed: 20 minutes.

One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as (a + b)(x + y) and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding (a + b + c)(x + y). (Common Core State Standards, p.4)

Mathematical and Scientific concepts have many representations; words, diagrams, algebraic symbols, tables, graphs and so on. It is important for students to learn to 'speak' these representations fluently and to learn to translate between them. The activity in **Handout 9** is intended to encourage students to discuss connections between verbal, numeric, spatial and algebraic representations. For the following activity, participants should work in pairs or threes.

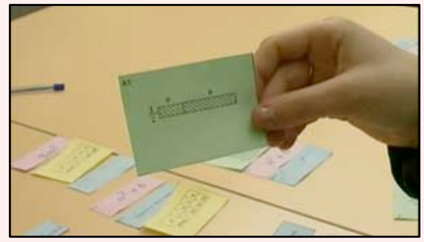
- Cut out the set of cards on **Handout 9.**
- Take it in turns to match Card Set A: *algebra expressions* with the Card Set B: *verbal descriptions*. Place pairs of cards side-by-side, face up on the table. If you find cards are missing, create these for yourself.
- Next, match Card set C: *tables* to the cards that you have already matched. You may find that a table matches more than one algebra expression. How can you convince yourself or your students that this will always be true, whatever the value for *n*?
- Next, match Card set D: *areas* to those cards that have already been grouped together. How do these cards help you to explain why different algebra expressions are equivalent? Discuss the difficulties that your students would have with this task.

The final matching may be made into a poster, as has been done here.



The next activity encourages participants to compare their own thinking with an episode of learning from the classroom. The students on the 5-minute video clip are all low attaining 16-17 years old who have had very little understanding of algebra previously.

- Watch **Activity E Video: 'Algebraic Expressions Lesson'**
- What difficulties do the students have while working on this task?
- How is the teacher helping students?



Finally, participants may begin to consider how this type of activity can be applied to representations that they teach.

- Which representations do you use most often in your classroom?
- Devise your own set of cards that will help your students translate between the different representations that you are teaching.

Handout 9: Interpreting multiple representations

Each group of students is given a set of cards. They are invited to sort the cards into sets, so that each set of cards have equivalent meaning. As they do this, they have to explain how they know that cards are equivalent. They also construct for themselves any cards that are missing. The cards are designed to force students to discriminate between commonly confused representations.

Card Set A: Expressions

E1 $\frac{n+6}{2}$	E2 $3n^2$
E3 $2n+12$	E4 $2n+6$
E5 $2(n+3)$	E6 $\frac{n}{2}+6$
E7 $(3n)^2$	E8 $(n+6)^2$
E9 $n^2+12n+36$	E10 $3+\frac{n}{2}$
E11 n^2+6	E12 n^2+6^2
E13	E14

Card Set B: Words

W1 Multiply n by two, then add six.	W2 Multiply n by three, then square the answer.
W3 Add six to n then multiply by two.	W4 Add six to n then divide by two.
W5 Add three to n then multiply by two.	W6 Add six to n then square the answer.
W7 Multiply n by two then add twelve.	W8 Divide n by two then add six.
W9 Square n, then add six	W10 Square n, then multiply by nine
W11	W12
W13	W14

Card Set C: Tables

T1

n	1	2	3	4
Ans	14	16	18	20

T2

n	1	2	3	4
Ans			81	144

T3

n	1	2	3	4
Ans		10	15	22

T4

n	1	2	3	4
Ans	3		27	48

T5

n	1	2	3	4
Ans			81	100

T6

n	1	2	3	4
Ans		10	12	14

T7

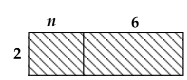
n	1	2	3	4
Ans		4		5

T8

n	1	2	3	4
Ans	6.5	7	7.5	8

Card Set D: Areas

A1



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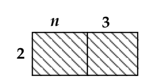

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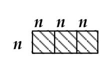

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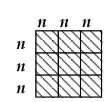

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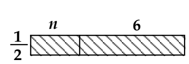

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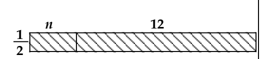

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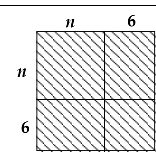

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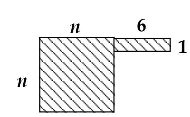

Activity F: Evaluating mathematical statements

Time needed: 20 minutes.

> *Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions.* (Common Core State Standards, p.6)

Students that are actively learning are constantly challenging hypotheses and critiquing the reasoning of others. The activities considered here are all designed to encourage these practices.

Ask participants to work together in groups of two or three using the activity on **Handout 10**.

> In this activity, you are given a collection of statements.
>
> - Decide on the validity of each statement and give explanations for your decisions. Your explanations will involve generating examples and counterexamples to support or refute the statements.
> - In addition, you may be able to add conditions or otherwise revise the statements so that they become 'always true'.
> - Create some statements that will create a stimulating discussion in your classroom.

This kind of activity is very powerful. The statements may be prepared to encourage students to confront and discuss common misconceptions or errors. The role of the teacher is to prompt students to offer justifications, examples, and counter-examples. For example:

Pay rise:
"OK you think it is sometimes true, depending on what Max and Jim earn. Can you give me a case where Jim gets the bigger pay rise? Can you give me an example where they both get the same pay rise?"

Area and perimeter:
"Can you give me an example of a cut that would make the perimeter bigger and the area smaller?"
"Suppose I take a bite out of this triangular sandwich. What happens to its area and perimeter?"

Right angles:
"Can you prove this is always true?"

Bigger fractions:
"You think this is always true? Can you draw me a diagram to convince me that this is so?"
"What happens when you start with a fraction greater than one?"

Handout 10: Evaluating mathematical statements

Pay rise Max gets a pay rise of 30%. Jim gets a pay rise of 25%. So Max gets the bigger pay rise.	**Sale** In a sale, every price was reduced by 25%. After the sale every price was increased by 25%. So prices went back to where they started.
Area and perimeter When you cut a piece off a shape you reduce its area and perimeter.	**Right angles** A pentagon has fewer right angles than a rectangle.
Birthdays In a class of ten students, the probability of two students being born on the same day of the week is one.	**Lottery** In a lottery, the six numbers 3, 12, 26, 37, 44, 45 are more likely to come up than the six numbers 1, 2, 3, 4, 5, 6.
Bigger fractions If you add the same number to the top and bottom of a fraction, the fraction gets bigger in value.	**Smaller fractions** If you divide the top and bottom of a fraction by the same number, the fraction gets smaller in value.
Square roots The square root of a number is less than or equal to the number	**Series** If the limit of the sequence of terms in an infinite series is zero, then the sum of the series is zero.

Activity G: Exploring the structure of problems

Time needed: 20 minutes.

> *Mathematically proficient students look closely to discern a pattern or structure.*
> (Common Core State Standards, p.6)

In many Mathematics Classrooms, students are presented with a variety of situations and problems. When a problem is solved, the temptation is to move to the next one, rather than to generalize the problem and explore its structure. Success is often related to the number of questions 'done' rather than the depth of understanding developed.

In this activity, we show two ways in which a deeper understanding may be achieved.

Students modifying a given problem.

Handout 11 shows one way in which a problem may be generalized by erasing the numbers in the problem, first one at a time, then two at a time, and exploring the relationships between the variables.

- Choose a problem from a textbook, or invent one of your own and try out this process for yourself.

Students creating their own problem situations

When students are offered the opportunity to *create* problems they may look at its structure differently. Here we ask students to work in pairs. Each creates a problem for the other to solve. **Handout 12** shows how this process can encourage students to consider the 'doing' and 'undoing' structures within mathematical situations. For example, one student might draw a circle and calculate its area. This student is then asked to pass the result to a neighbor, who must now try to reconstruct the circle from the given area. Both students then collaborate to see where mistakes have arisen.

- Think of some more occasions when students can engage in these 'doing' and 'undoing' processes.
- Which process in each pair is easier? This process may be the most helpful one to use for the problem creator.
- Which processes do not have a unique inverse? This will result in both students in a pair obtaining different correct work!

Handout 11: Students modifying a given problem

Here is a typical word problem from a textbook.

The candles problem

A student wants to earn some money by making and selling candles.
Suppose that she can make 60 candles from a $50 kit, and that these will each be sold for $4.
How much profit will she make?

After answering such a question, we might explore its structure and attempt some generalizations.
First remove all the numbers from the problem:

The cost of buying the kit: (This includes the molds, wax and wicks.)	$	*k*: 50	
The number of candles that can be made with the kit:		*n*: 60	candles
The price at which he sells each candle:	$	*s*: 4	per candle
Total profit made if all the candles are sold:	$	*p*: 190	

Now we can ask the following, first using numerical values, then using variables:

1. *How did we calculate the profit p using the given values of k, n, and s?*
   Would your method change if the values of k, n, and s were different?
2. *Write in the profit and erase one of the other values: the selling price of each candle, s.*
   How can you figure out the value of s from the remaining values of k, n and p?
   Repeat, but now erase the value of a different variable and say how it may be reconstructed from the remaining values.
3. *Suppose you didn't know either of the values of n and p, but you knew the remaining values.*
   How will the profit depend on the number of candles made? Plot a graph.
   Repeat for other pairs of variables.

Write down four general formulas showing the relationships between the variables.

p = ………………… *s* = ………………… *n* = ………………… *k* = …………………

Handout 12: Students creating problems for each other

Ask students to work in pairs. Each creates a problem for the other to solve.

Doing: The problem poser…	**Undoing: The problem solver…**
• generates an equation step-by-step, starting with, say, $x = 4$ and 'doing the same to both sides'	• solves the resulting equation: $\frac{10x+9}{8} - 7 = -0.875$
• draws a rectangle and calculates its area and perimeter.	• tries to draw a rectangle with the given area and perimeter.
• writes down an equation of the form $y=mx+c$ and plots a graph.	• tries to find an equation that fits the resulting graph.
• expands an algebraic expression such as $(x+3)(x-2)$	• factorizes the resulting expression: • $x^2 + x - 6$
• writes down a polynomial and differentiates it	• integrates the resulting function
• writes down five numbers and finds their mean, median, range	• tries to find five numbers with the given mean, median and range.

Activity H: Plan a lesson, teach it and reflect on the outcomes

Time needed:
15 minutes discussion before the lesson.
1 hour for the lesson.
15 minutes after the lesson.

Planning a lesson

Choose one of the Concept Lessons in this unit that you feel would be appropriate for your class.

Discuss when/how you will:

- Organize the preliminary assessment.
- Plan the classroom layout and the resources needed.
- Introduce the problem to students.
- Explain to students how you want them to work together.
- Challenge/assist students that find the activity straightforward/ difficult.
- Help them share and learn from each other.
- Draw out the important concepts from the lesson.

If you are working on this module with a group, it will be helpful if each participant chooses the same problem, as this will facilitate the follow-up discussion.

Reflection on a lesson

Now you have taught the lesson, it is time to reflect on what happened.

- What range of responses did students have to the initial assessment task?
- Which feedback questions did you prepare for the class? Did you add any of your own feedback questions?
- Did the activities in the lesson enable students to discuss their main conceptual difficulties?
- What was your role during the main activity?
- What support and guidance did you need to provide? Why was this?
- What do you think students learned from this lesson?

MAP Lessons for Formative Assessment of Concept Development

The MAP lessons for formative assessment of concept development produced, to date, are:

High School
Solving Linear Equations in Two Variables *
Evaluating Statements About Length and Area *
Evaluating Statements About Enlargements (2D & 3D) *
Calculating Volumes of Compound Objects *
Sorting Equations and Identities *
Graphing Inequalities in Two Variables *
Interpreting Algebraic Expressions *
Representing and Combining Transformations *
Forming Quadratics *
Finding Equations of Parallel and Perpendicular Lines *
Ferris wheel
Representing Data Using Frequency Graphs *
Representing Data Using Box Plots
Rational and Irrational Numbers 1
Analyzing Congruency Proofs
Rational and Irrational Numbers 2
Modelling Conditional Probabilities 2 *
Equations of Circles 1 *
Equations of Circles 2
Comparing Investments
Sectors of circles
Manipulating Radicals
Representing polynomials
Creating and Solving Equations
Manipulating Polynomials
Functions and Everyday Situations
2D representations of 3D objects

Middle School
Increasing and Decreasing Quantities by a Percent *
Applying Angle Theorems *
Evaluating Statements About Probability *
Positive and Negative Numbers
Steps to Solving Equations
Interpreting Distance-Time Graphs *
Modeling Situations With Linear Equations *
Estimating Length, Using Scientific Notation
Lines and Linear Equations
Repeating decimals
The Pythagorean Theorem: Square areas *
Classifying Solutions to Systems of Equations
Solving Linear Equations in One Variable

Lessons marked * are available to all from the site map.mathshell.org. The other lessons will be released in Q2 2012, and are available in preview form, on request, to Mathematics Development Collaborative partners.